LOCAL HEATING OF A CYLINDER WITH AN INCLUSION

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UDC 536.24

Exact and approximate solutions are presented for the stationary heat-conduction problem for a cylinder with a foreign inclusion for a discontinuous boundary condition of the first kind. Limits of applicability are set for the approximate solutions.

Glued connections, weld butts of solid bodies, bonding plate and shell elements are modeled by different fine inclusions. To describe the thermal processes in such bodies, the model of nonideal contact between solids [1-4] or a continual model for bodies with fine inclusions [5, 6] are used. We analyze the limits of applicability of these models in the simplest example. To this end we consider an infinite cylinder of radius r_2 that contains a foreign inclusion in the form of a coaxial thin cylindrical shell of thickness 2h and middle surface radius R. A discontinuous boundary condition of the first kind is given on the cylinder side surface $r = r_2$.

We have the heat-conduction equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\lambda(r)\frac{\partial t}{\partial r}\right]+\lambda(r)\frac{\partial^2 t}{\partial z^2}=0 \tag{1}$$

and boundary conditions

$$t|_{r=r_2}=t_0N(z),\ t|_{r=0}\neq\infty,\ \left\{t,\ \frac{\partial t}{\partial z}\right\}\Big|_{|z|\to\infty}\to 0, \tag{2}$$

to determine the stationary temperature field that occurs here, where $N(z) = S_-(z + d) - S_+(z - d)$, and $S_\pm(\xi)$ are asymmetric unit functions [7].

The heat-conduction coefficient $\lambda(r)$, which is a function of the radial coordinate r, is represented in the form

$$\lambda(r)=\lambda_1+(\lambda_0-\lambda_1)N_*(r,\ h), \tag{3}$$

where λ_0, λ_1 are, respectively, the heat-conduction coefficients of the inclusion and the main material, and $N_*(r, h) = S_-(r - R + h) - S_+(r - R - h)$.

Substituting (3) into (1) and executing the manipulations needed, we find

$$\Delta t=(1-K_\lambda^{-1})\left[\frac{\partial t}{\partial r}\Big|_{r=R-h+0}\delta_-(r-R+h)-\frac{\partial t}{\partial r}\Big|_{r=R+h-0}\delta_+(r-R-h)\right], \tag{4}$$

where $K_\lambda = \lambda_1/\lambda_0$ is the criterion characterizing the relative heat conduction of the main material with respect to the inclusion [8]

$$\delta_\pm(\xi)=S_\pm'(\xi),\quad \Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial}{\partial z^2}.$$

Applying a Fourier transform in z to (4) and (2), we obtain

$$\frac{d^2\bar t}{dr^2}+\frac{1}{r}\frac{d\bar t}{dr}-\xi^2\bar t=f(r), \tag{5}$$

Institute of Mechanics and Mathematics Problems, Academy of Sciences of the Ukrainian SSR, Lvov. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 53, No. 4, pp. 648-654, September, 1987. Original article submitted June 17, 1986.

 0022-0841/87/5304-1204$12.50

$$\bar{t}|_{r=r_2} = t_0 \sqrt{\frac{2}{\pi}} \frac{\sin \zeta d}{\zeta}, \quad \bar{t}|_{r=0} \neq \infty, \tag{6}$$

where

$$f(r) = (1 - K_\lambda^{-1}) \left[\frac{d\bar{t}}{dr} \bigg|_{r=R-h+0} \delta_-(r - R + h) - \frac{d\bar{t}}{dr} \bigg|_{r=R+h-0} \delta_+(r - R - h) \right].$$

The solution of the boundary-value problem (5) and (6) has the form

$$t(r, \zeta) = t_0 \sqrt{\frac{2}{\pi}} \frac{\sin \zeta d \Omega(r, |\zeta|)}{\zeta \Omega(r_2, |\zeta|)}, \tag{7}$$

where

$$\Omega(r, |\zeta|) = I_0(|\zeta| r) + (1 - K_\lambda^{-1}) |\zeta| \langle (R-h) I_1(\zeta_-) \Omega_1(r, |\zeta|, R-h) \times$$
$$\times \{ S_-(r - R + h) - (R+h) \{ I_1(\zeta_+) + (K_\lambda - 1) \zeta_- I_1(\zeta_-) [I_1(\zeta_+) K_0(\zeta_-) +$$
$$+ K_1(\zeta_+) I_0(\zeta_-)] \} \Omega_1(r, |\zeta|, R+h) S_+(r - R - h) \rangle;$$
$$\Omega_1(r, |\zeta|, \xi) = I_0(|\zeta| r) K_0(|\zeta| \xi) - K_0(|\zeta| r) I_0(|\zeta| \xi); \quad |\zeta|_\pm = |\zeta|(R \pm h);$$

$I_\nu(\xi)$, $K_\nu(\xi)$ are modified Bessel functions ($\nu = 0, 1$).

Using the inversion formula and the convolution theorem [9], we find the solution of the heat-conduction problem for the cylindrical body under consideration with the inclusion in the form

$$t_T(r, z) = \frac{2t_0}{\pi} \int_0^\infty \frac{\Omega(r, \zeta)}{\zeta \Omega(r_2, \zeta)} \sin \zeta d \cos \zeta z d\zeta. \tag{8}$$

When the inclusion thickness 2h is small compared with the other dimensions, we apply the approach proposed in [5, 6]. To do this we introduce the reduced heat conduction of the inclusion $\Lambda_0 = 2\lambda_0 h$ and we use the Dirac delta-function approximation [5]

$$\delta(r - R) = \lim_{h \to 0} \frac{N_*(r, h)}{2h}$$

and the identity

$$f(r) \delta(r - R) = \frac{1}{2} [f(R+0) + f(R-0)] \delta(r - R).$$

Then (1) is rewritten in the following manner

$$\Delta t = -\gamma \left[\left(\frac{\partial^2 t}{\partial z^2} \right)^* \bigg|_{r=R} \delta(r - R) + \frac{R}{r} \left(\frac{\partial t}{\partial r} \right)^* \bigg|_{r=R} \delta'(r - R) \right], \tag{9}$$

where

$$\gamma = \Lambda_0 (1 - K_\lambda^{-1}) \lambda_1^{-1}; \quad \left(\frac{\partial t}{\partial r} \right)^* \bigg|_{r=R} = \frac{1}{2} \left[\frac{\partial t}{\partial r} \bigg|_{r=R+0} + \frac{\partial t}{\partial r} \bigg|_{r=R-0} \right];$$

$$\left(\frac{\partial^2 t}{\partial z^2} \right)^* \bigg|_{r=R} = \frac{1}{2} \left[\frac{\partial^2 t}{\partial z^2} \bigg|_{r=R+0} + \frac{\partial^2 t}{\partial z^2} \bigg|_{r=R-0} \right].$$

By using the Fourier integral transform, the solution of the boundary-value problem (9) and (2) takes the form

$$t(r, z) = \frac{2t_0}{\pi} \int_0^\infty \frac{\Phi(r, \zeta)}{\zeta \Phi(r_2, \zeta)} \sin \zeta d \cos \zeta z d\zeta, \tag{10}$$

where

$$\Phi(r,\ \zeta)=\left(1+\frac{\gamma^2\zeta^2}{4}\right)I_0(\zeta r)+\gamma\zeta^2R\,[I_0(\zeta r)\,\varphi_1(\zeta)-K_0(\zeta r)\,\varphi_2(\zeta)]\,S(r-R);$$

$$\varphi_1(\zeta)=I_0(\zeta R)\,K_0(\zeta R)-I_1(\zeta R)\,K_1(\zeta R)-\frac{\gamma}{2R};$$

$$\varphi_2(\zeta)=I_0^2(\zeta R)+I_1^2(\zeta R);$$

$S(r-R)=\begin{cases}1, & r>R\\ 0{,}5, & r=R\\ 0, & r<R\end{cases}$ is the symmetric unit Heaviside function.

A solution of the one-dimensional nonstationary heat-conduction problem for a system of two plates and a two-dimensional problem for a space with a foreign cylindrical inclusion has been obtained in [4] in an approximate formulation in the case of a small intermediate layer thickness. Conditions of nonideal thermal contact are considered here.

To compare with the solutions of the form (8) and (10) obtained, we present the solution of the heat-conduction problem for a cylinder with a foreign inclusion by considering that conditions of nonideal thermal contact hold here [1-4].

We assume that two cylindrical bodies are connected by a butt by using an intermediate layer of thickness 2h and heat conduction λ_0. The temperature of the outer cylinder is denoted by t_2 in the domain $R \le r \le r_2$ and the inner by t_1 $(0 \le r < R)$. The boundary and contact conditions have the form

$$\left[\Lambda_0\frac{\partial^2(t_1+t_2)}{\partial r^2}-2\lambda_1\left(\frac{\partial t_1}{\partial r}-\frac{\partial t_2}{\partial r}\right)\right]\Bigg|_{r=R}=0, \tag{11}$$

$$\left[\Lambda_0\frac{\partial^2(t_1-t_2)}{\partial r^2}-6\lambda_1\left(\frac{\partial t_1}{\partial r}+\frac{\partial t_2}{\partial r}\right)\right]\Bigg|_{r=R}=\frac{12}{r_0}(t_1-t_2)|_{r=R}, \tag{12}$$

$$t_1|_{r=0}\neq\infty,\quad t_2|_{r=r_2}=t_0N(z),$$

where $r_0 = 2h/\lambda_0$ is the thermal resistivity.

Setting $\Lambda_0 \to 0$ in the conditions (11), we will have in place of (11)

$$\left(\frac{\partial t_1}{\partial r}-\frac{\partial t_2}{\partial r}\right)\Bigg|_{r=R}=0,\quad (t_2-t_1)|_{r=R}=\lambda_1 r_0\,\frac{\partial t_1}{\partial r}\Bigg|_{r=R}. \tag{13}$$

Therefore, each of the conditions (11) and (13) is an appropriate extension of the ideal thermal contact conditions.

The t_1 and t_2 satisfy stationary heat-conduction conditions in the domans $0 \le r < R$, $R < r \le r_2$

$$\Delta t_j=0\quad (j=1,\ 2). \tag{14}$$

By using the Fourier integral transform in z, the expressions for the transforms $\bar t_1$ and $\bar t_2$, satisfying the transformed boundary and contact conditions (11) and (12), will be represented in the form

$$\bar t_1(r,\ \zeta)=t_0\sqrt{\frac{2}{\pi}}\,\frac{\sin\zeta d I_0(|\zeta|\,r)}{\zeta\Delta(r_2,\ |\zeta|)},$$

$$\bar t_2(r,\ \zeta)=t_0\sqrt{\frac{2}{\pi}}\,\frac{\sin\zeta d\Delta(r,\ |\zeta|)}{\zeta\Delta(r_2,\ |\zeta|)}, \tag{15}$$

where

$$\Delta(r,\ |\zeta|)=2\zeta^2K_0(|\zeta|\,r)\left[\Lambda_0\left(\Lambda_0\zeta^2+\frac{12}{r_0}\right)I_0(|\zeta|\,R-12\lambda_1^2I_1^2(|\zeta|\,R)\right]-$$

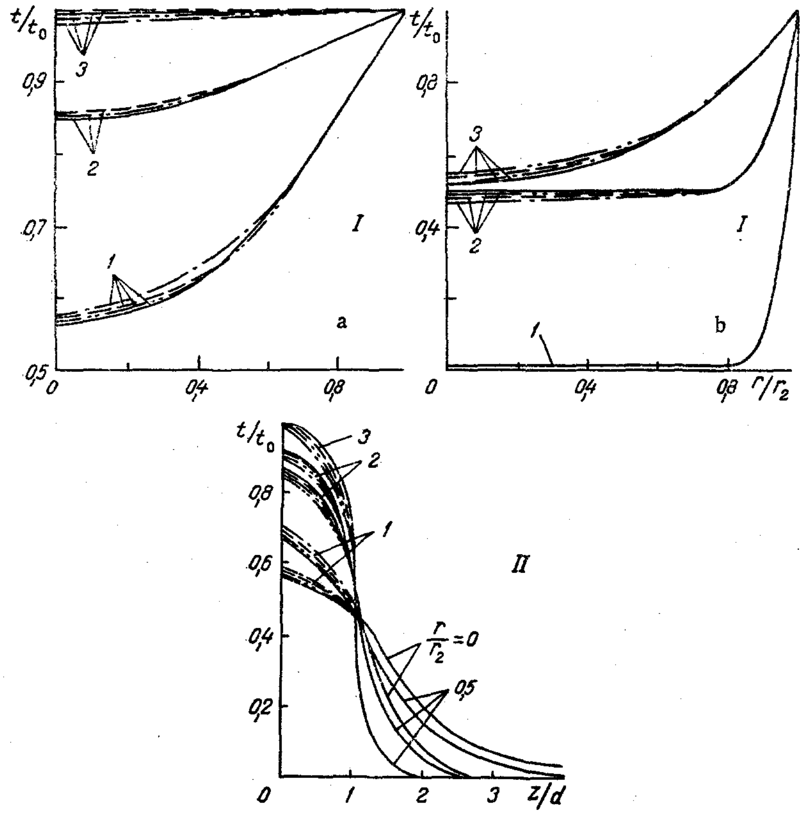

Fig. 1. Dimensionless temperature distribution for $R/r_2 = 0.5$, $h/r_2 = 10^{-2}$: I) along the radial coordinate: a) $z/d = 0$, 1) $d/r_2 = 0.5$; 2) $d/r_2 = 1$; 3) $d/r_2 = 2$, b) $z/d = 1$; 1) $d/r_2 = 2$; 2) $d/r_2 = 1$; 3) $d/r_2 = 0.5$; II) along the axial coordinates: 1) $d/r_2 = 0.5$; 2) $d/r_2 = 1$; 3) $d/r_2 = 2$.

$$- 2I_0(|\zeta| r)\Big[12\lambda_1^2\zeta^2 K_1(|\zeta| R)\, I_1(|\zeta| R) + + \Lambda_0\zeta^2\left(\Lambda_0\zeta^2 + \frac{12}{r_0}\right) I_0(|\zeta| R)\, K_0(|\zeta| R) + 3\lambda_1\Lambda_0\frac{\zeta^2}{R} + \frac{\lambda_1}{R}\left(\Lambda_0\zeta^2 + \frac{R}{r_0}\right)\Big]. \quad (16)$$

In the case of the contact conditions (13), the expression $\Delta(r, |\zeta|)$ is written in the following way

$$\Delta(r, |\zeta|) = I_0(|\zeta| r) + \frac{2h\zeta^2 R}{K_\lambda}\,[I_1(|\zeta| R)\, K_0(|\zeta| r) + K_1(|\zeta| R)\, I_0(|\zeta| r)]. \quad (17)$$

We obtain respectively for t_1 and t_2

$$t_1(r, z) = \frac{2t_0}{\pi}\int_0^\infty \frac{I_0(|\zeta| r)}{\zeta\Delta(r_2, \zeta)}\sin\zeta d\cos\zeta z d\zeta, \quad (18)$$

$$t_2(r, z) = \frac{2t_0}{\pi}\int_0^\infty \frac{\Delta(r, \zeta)}{\zeta\Delta(r_2, \zeta)}\sin\zeta d\cos\zeta z d\zeta; \quad (19)$$

where $\Delta(r, \zeta)$ is represented by (16) or (17) depending on the kind of contact conditions.

Using the ES-1060 electronic computer, numerical investigations were performed for the temperature field by means of (8), (10), (18) and (19), where a set of programs was compiled which was approved for the OS ES. For the calculations we took $\lambda_1/\lambda_0 = 0.25$.

Results of investigating the dimensionless temperature t/t_0 are represented in Fig. 1. The solid lines correspond to the exact solution determined by (8), and the dashes to the temperature calculated by means of (10), the dash-dot curves correspond to the case of general contact conditions (11) and the dash with two dots to $\Lambda_0 \to 0$, i.e., the thermal contact conditions taken in the form (13).

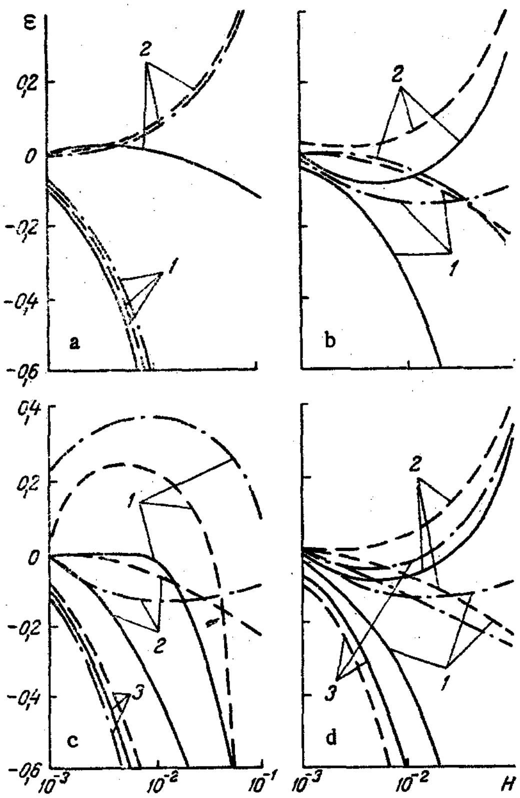

Fig. 2. Change in relative error ε as a function of the parameter H: a) $d/r_2 = 1$; $R/r_2 = 0.5$; $z/d = 0$, 1) $r/r_2 = 0$; 2) $r/r_2 = 0.9$; b) $d/r_2 = 1$; $R/r_2 = 0.5$; $r/r_2 = 0.5$, 1) $z/d = 0$; 2) $z/d = 1$, c) $d/r_2 = 1$; $r/r_2 = 0.5$; $z/d = 0$, 1) $R/r_2 = 0.25$; 2) $R/r_2 = 0.5$; 3) $R/r_2 = 0.75$, d) $R/r_2 = 0.5$; $r/r_2 = 0.5$; $z/d = 0$, 1) $d/r_2 = 0.5$; 2) $d/r_2 = 1$; 3) $d/r_2 = 2$.

From the results of the investigations the following can be noted: The width of the local heating zone exerts noticeable influence on the nature of the temperature distribution. Thus, for instance, the maximal difference between the values of the dimensionless temperature at the side surface in the local heating zone and at the point $r_1/r_2 = 0.5$ is 0.02 for $d/r_2 = 2$ while this difference is 0.32 for $d/r_2 = 0.5$. The maximal divergence between the temperature values determined by the fine inclusion model and with the nonideal thermal contact taken into account is 3.9% but the difference between the exact solution and a solution of the form (10) is not more than 2%. It is established that a change in the parameters h/r_2 (in the 10^{-3}-10^{-2} band) and R/r_2 (in the 0.1-0.75 band) exerts no substantial influence on the difference between the temperature values expressed by (8), (10), (18) and (19). In the case of nonideal thermal contact conditions (11), for selected values of the parameters the temperature jump at the point $r = R$ is manifest only in the second symbol after the decimal for $d/r_2 = 0.5$; later as the parameter d/r_2 characterizing the magnitude of the local thermal action zone increases, this difference is observed only in the fourth place after the decimal.

The solid lines in Fig. 2 correspond to values of the relative error $\varepsilon = (t_T - t)/t_T \cdot 100\%$ between the exact solution and the solution determined by means of the model of fine inclusions; the dashed and dash-dot curves, respectively, correspond to the errors between the exact solution and the solutions that correspond to the nonideal thermal contact conditions (11) and (13).

As computations show, the error in determining the temperature by using approximate approaches will grow as the quantity R/r_2 increases, i.e., the parameter characterizing the remoteness of the foreign inclusion from the local thermal action zone. For $H \leq 10^{-2}$ the relative error by all the approximate approaches does not exceed 1%. The numerical investigations performed show that as the parameter H grows the error in determining the temperature by all the approximate schemes considered here will grow.

NOTATION

t, temperature; r, z, radial and axial coordinates; 2h, thickness of the inclusion; and $\delta(\xi)$, Dirac delta function.

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NUMERICAL-ANALYTICAL METHOD OF SOLVING THE NONLINEAR HEAT-CONDUCTION PROBLEM FOR A DOUBLY CONNECTED VARIABLE-THICKNESS PLATE

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UDC 536.21

The method permits construction of an approximate solution in an analytically closed form on each of the radial rays into which the plate domain is separated.

1. We have an isotropic plate whose external L_1 and internal L_2 contours are described by equations in a dimensionless polar coordinate system r, Θ:

$$r = r_\nu [1 + \varepsilon_\nu (\cos m_\nu \Theta + a_\nu \cos 2m_\nu \Theta)], \quad \nu = 1,\ 2, \qquad (1)$$

where $\nu = 1$ corresponds to the external and $\nu = 2$ to the internal contours, r_ν, ε_ν, m_ν, a_ν are parameters, and $|\varepsilon_\nu| < 1$, $|a_\nu| < 1$, $r_\nu < 1$.

The plate thickness h(r, Θ) varies according to the law

$$h(r,\ \Theta) = H(r)\,P(\Theta), \qquad (2)$$

where H(r) and P(Θ) are given functions.

Boundary conditions of the first kind are satisfied on the side surfaces, i.e.,

$$T = T_\nu(\Theta) \text{ on } L_\nu,\ \nu = 1,\ 2. \qquad (3)$$

Here the period of the function $T_\nu(\Theta)$ equals $2\pi/k_0$, where k_0 is a positive integer.

Let us consider the foundation of the plate heat insulated. The thermal characteristics of the material depend on the temperature.

The heat-conduction differential equation for a function of the temperature T has the form [1]

$$\frac{\partial}{\partial r}\left(rh\lambda \frac{\partial T}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \Theta}\left(h\lambda \frac{\partial T}{\partial \Theta}\right) = 0. \qquad (4)$$

Here $\lambda = \lambda(T)$ is the heat-conduction coefficient.

Introducing the Kirchhoff variable

Saratov Polytechnic Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 53, No. 4, pp. 654-658, October, 1987. Original article submitted July 14, 1986.